The University of Chicago

MOTION OF A FLEXIBLE CABLE IN A VERTICAL PLANE

A DISSERTATION

SUBMITTED TO THE FACULTY OF THE OGDEN GRADUATE SCHOOL OF SCIENCE IN CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY, DEPARTMENT OF MATHEMATICS

BY

ELMER DANIEL GRANT

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LANCASTER, PA.

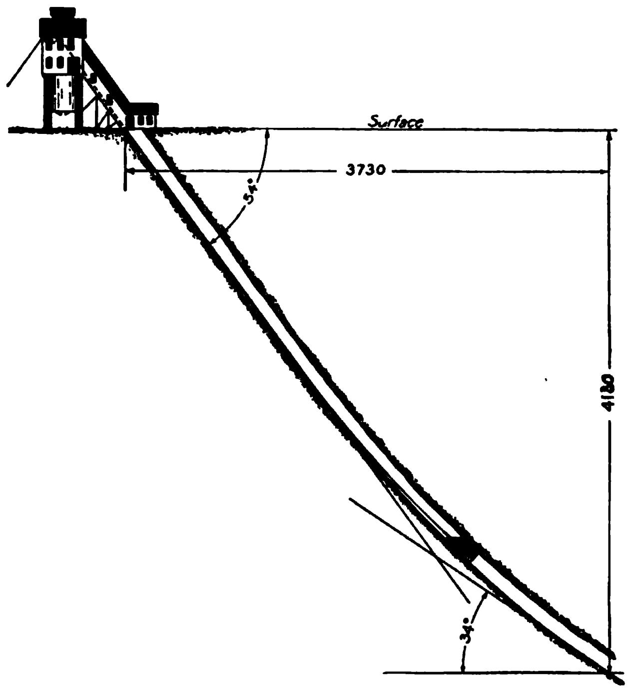
MOTION OF A FLEXIBLE CABLE IN A VERTICAL PLANE

By ELMER D. GRANT

Introduction

This general problem of the motion of a flexible cable in a vertical plane has been discussed by Routh, Appell, Resal, Floquet, Leauté and others, but the following discussion will deal with the particular problem of uniformly accelerated motion of a flexible cable. Part I is devoted to the inextensible cable, while the case of the extensible cable is treated in Part II.

The problem was suggested by observing the fluctuations in the movements of the steel cables used in hoisting loads of copper rock in the copper mines of Michigan. In some of these mines the shaft is in the shape of a common catenary, the inclination at the surface being about 54°, while about 4,200 feet farther down the inclination has decreased to 34°. The accompanying diagram illustrates the conditions, which are approximately the same as the numerical data in the catenary drawn in Part III.



The question arose, what would be the shape of the curve assumed by the cable when subject to an uniform acceleration? The speed attained is frequently as high as 50 miles per hour.

PART I. EQUATIONS OF MOTION OF A FLEXIBLE, INEXTENSIBLE CABLE IN A VERTICAL PLANE

A. THE COMMON CATENARY

1. *Intrinsic Equation*

For comparison with other cases it will be necessary to have the equations of the common catenary. We derive first the intrinsic equation of the curve assumed by a flexible, inextensible cable at rest. The inclination will be reckoned from the horizontal. In Fig. 1 let s be the length of the arc OP; ϕ and ϕ_0 the inclinations at P and O, respectively; and T and T_0 the tensions at P and O, respectively. Then, if γ equals the weight per unit length of the cable, we have

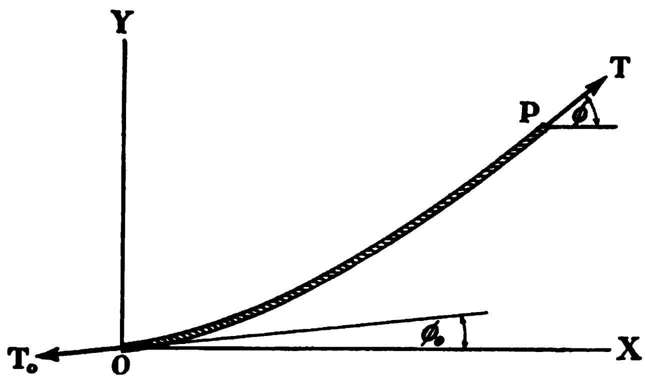


FIG. 1

$$\Sigma X = T\cos\phi - T_0\cos\phi_0 = 0, \tag{1}$$

$$\Sigma Y = T\sin\phi - T_0\sin\phi_0 - \gamma s = 0. \tag{2}$$

Therefore, from equation (1) the tension at any point P is

$$T = T_0\cos\phi_0\sec\phi. \tag{3}$$

On eliminating T from (1) and (2), we get the intrinsic equation of the catenary

$$s = \frac{T_0}{\gamma}\cos\phi_0(\tan\phi - \tan\phi_0). \tag{4}$$

Since the radius of curvature is $\rho = ds/d\phi$, we have the relation

$$\rho = \frac{T_0}{\gamma}\cos\phi_0\sec^2\phi. \tag{5}$$

If the inclination is reckoned from the tangent at O instead of from the horizontal, equations (4) and (5) become

$$s = \frac{T_0}{\gamma}\tan\phi, \text{ and } \rho = \frac{T_0}{\gamma}\sec^2\phi.$$

If equation (4) is expanded into a series in $\phi - \phi_0$, we get

$$s = \frac{T_0}{\gamma}\left\{\begin{aligned} &\sec\phi_0(\phi-\phi_0) + 2\sec^2\phi_0\sin\phi_0(\phi-\phi_0)^2/2! \\ &+ \sec^3\phi_0(4\sin^2\phi_0 + 2)(\phi-\phi_0)^3/3! \\ &+ \sec^4\phi_0\sin\phi_0(8\sin^2\phi_0 + 16)(\phi-\phi_0)^4/4! \\ &+ \sec^5\phi_0(16\sin^4\phi_0 + 88\sin^2\phi_0 + 16)(\phi-\phi_0)^5/5! + \cdots \end{aligned}\right\}. \tag{6}$$

2. *Parametric Representation*

A parametric representation of the catenary in terms of the parameter ϕ may be found by the relations

$$dx = \cos\phi\, ds, \quad \text{and} \quad dy = \sin\phi\, ds.$$

Therefore

$$\left.\begin{aligned} dx &= \frac{T_0}{\gamma}\cos\phi_0\sec\phi\, d\phi, \\ dy &= \frac{T_0}{\gamma}\cos\phi_0\sec\phi\tan\phi\, d\phi. \end{aligned}\right\} \tag{7}$$

By integration, we obtain

$$\left.\begin{aligned} x &= \frac{T_0}{\gamma}\cos\phi_0\log_e\frac{\sec\phi + \tan\phi}{\sec\phi_0 + \tan\phi_0} = \frac{T_0}{\gamma}\cos\phi_0(gd^{-1}\phi - gd^{-1}\phi_0), \\ y &= \frac{T_0}{\gamma}\cos\phi_0(\sec\phi - \sec\phi_0). \end{aligned}\right\} \tag{8}$$

In equations (8) $gd^{-1}\phi$ represents the anti-gudermannian of ϕ.

If the origin is taken at the lowest point, (8) becomes

$$\begin{aligned} x &= \frac{T_0}{\gamma}\log_e(\sec\phi + \tan\phi) = \frac{T_0}{\gamma}gd^{-1}\phi, \\ y &= \frac{T_0}{\gamma}(\sec\phi - 1). \end{aligned}$$

3. *Equation in Rectangular Coördinates*

To express x in terms of y, we solve the second equation of (8) for $\sec\phi$ and substitute in the first. This gives

$$x = \frac{T_0}{\gamma}\cos\phi_0\log_e\frac{\sec\phi_0(y + T_0/\gamma) + \sqrt{(y + T_0/\gamma)^2\sec^2\phi_0 - (T_0/\gamma)^2}}{T_0/\gamma(\sec\phi_0 + \tan\phi_0)}. \tag{9}$$

If $\phi_0 = 0$, this reduces to

$$x = \frac{T_0}{\gamma} \log_e \frac{y + T_0/\gamma + \sqrt{2yT_0/\gamma + y^2}}{T_0/\gamma}.$$

On solving (9) for y, we obtain

$$y = -\frac{T_0}{\gamma} + \frac{1}{2}\frac{T_0}{\gamma} \cos \phi_0[(\sec \phi_0 + \tan \phi_0)e^{\frac{x\gamma \sec \phi_0}{T_0}} + (\sec \phi_0 - \tan \phi_0)e^{-\frac{x\gamma \sec \phi_0}{T_0}}]. \quad (10)$$

If $\phi_0 = 0$, equation (10) reduces to

$$y = -\frac{T_0}{\gamma} + \frac{1}{2}\frac{T_0}{\gamma}[e^{\frac{x\gamma}{T_0}} + e^{-\frac{x\gamma}{T_0}}] = -\frac{T_0}{\gamma} + \frac{T_0}{\gamma} \cosh \frac{x\gamma}{T_0}.$$

On expanding (10) into a power series in x, we obtain

$$y = x \tan \phi_0 + \frac{x^2 \sec^2 \phi_0}{2!\,T_0/\gamma} + \frac{x^3 \sec^3 \phi_0 \sin \phi_0}{3!\,(T_0/\gamma)^2} + \frac{x^4 \sec^4 \phi_0}{4!\,(T_0/\gamma)^3} + \frac{x^5 \sec^5 \phi_0 \sin \phi_0}{5!\,(T_0/\gamma)^4} + \frac{x^6 \sec^6 \phi_0}{6!(T_0/\gamma)^5} + \cdots. \quad (11)$$

This series is convergent for all finite values of x. If $\phi_0 = 0$, the series contains only the even powers of x:

$$y = \sum_{n=1}^{n=\infty} \frac{x^{2n}}{(2n)!\,(T_0/\gamma)^{2n-1}}.$$

B. UNIFORMLY ACCELERATED MOTION

4. *Tension at any Point of the Curve*

For steady motion it is necessary and sufficient that for every point of the cable the velocity in a direction normal to the cable shall be zero. Appell has shown* that the

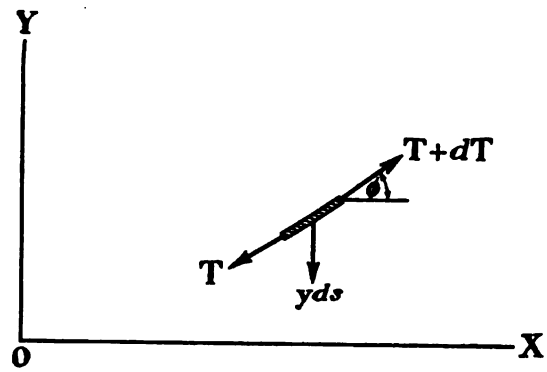


FIG. 2

* *Acta Mathematica*, Vol. XII, 1889, Art. 6, pp. 13, 14, 15.

necessary and sufficient condition that the normal velocity be zero at all points of the cable is that the tangential velocity of a point on the cable vary directly as the time; that is, the acceleration in the direction of the tangent is constant.

We therefore let the cable have a tangential velocity v and a uniform acceleration a along the curve. The motion will then be steady and the velocity will be constant with respect to T, ϕ and s.

Consider the forces (Fig. 2) acting on an element ds of the cable. Summing the tangential and normal components, we have

$$\Sigma T = dT - \gamma \sin \phi \, ds = a \frac{\gamma \, ds}{g}, \tag{12}$$

$$\Sigma N = T \, d\phi - \gamma \cos \phi \, ds = \frac{v^2}{\rho} \frac{\gamma \, ds}{g} = \frac{\gamma}{g} v^2 \, d\phi. \tag{13}$$

Equations (12) and (13) may be written as follows:

$$\frac{dT}{ds} = \gamma \left(\sin \phi + \frac{a}{g} \right), \tag{14}$$

$$\left(T - \frac{\gamma v^2}{g} \right) \frac{d\phi}{ds} = \gamma \cos \phi. \tag{15}$$

On dividing (14) by (15), we obtain

$$\frac{\dfrac{dT}{ds}}{T - \dfrac{\gamma v^2}{g}} = \left(\tan \phi + \frac{a}{g} \sec \phi \right) \frac{d\phi}{ds}. \tag{16}$$

On integrating (16), we obtain

$$\log_e \left(T - \frac{\gamma v^2}{g} \right) = \log_e \sec \phi + \frac{a}{g} \log_e (\sec \phi + \tan \phi) + \log_e c. \tag{17}$$

Therefore

$$\frac{T}{\gamma} - \frac{v^2}{g} = \frac{c}{\gamma} \sec \phi (\sec \phi + \tan \phi)^{a/g}. \tag{18}$$

Since $T = T_0$ when $\phi = \phi_0$, we have

$$\frac{c}{\gamma} = \frac{\left(\dfrac{T_0}{\gamma} - \dfrac{v^2}{g} \right) \cos \phi_0}{(\sec \phi_0 + \tan \phi_0)^{a/g}}. \tag{19}$$

The tension at any point is expressed as a function of ϕ in equation (18). This can be written in the form

$$T = \frac{\gamma v^2}{g} + \left(T_0 - \frac{\gamma v^2}{g} \right) \frac{\sec \phi}{\sec \phi_0} \left(\frac{\sec \phi + \tan \phi}{\sec \phi_0 + \tan \phi_0} \right)^{a/g}. \tag{20}$$

When expressed as a power series in a/g, this equation becomes

$$T = \frac{\gamma v^2}{g} + \left(T_0 - \frac{\gamma v^2}{g}\right)\frac{\sec\phi}{\sec\phi_0}\left[1 + \sum_{n=1}^{\infty}\frac{(a/g)^n}{n!}\left\{\log_e \frac{\sec\phi + \tan\phi}{\sec\phi_0 + \tan\phi_0}\right\}^n\right]. \quad (21)$$

If the velocity is constant, that is, $a = 0$, equation (20) becomes

$$T - \frac{\gamma v^2}{g} = \left(T_0 - \frac{\gamma v^2}{g}\right)\cos\phi_0 \sec\phi,$$

which is in the same form as equation (3) for the common catenary, the parameter T_0 being replaced by $T_0 - \gamma v^2/g$. The tension at any point is increased by $\gamma v^2/g$, the curve remaining the same as though the cable were at rest.* The term common catenary will therefore be used to designate the curve assumed by a uniform, inelastic, flexible cable either at rest or moving with a uniform velocity.

If T_c represent the tension at any point in the common catenary having the same T_0, ϕ_0, and v, then equation (21) can be written

$$T = T_c + \left(T_0 - \frac{\gamma v^2}{g}\right)\frac{\sec\phi}{\sec\phi_0}\sum_{n=1}^{\infty}\frac{(a/g)^n}{n!}\left\{\log_e \frac{\sec\phi + \tan\phi}{\sec\phi_0 + \tan\phi_0}\right\}^n. \quad (22)$$

5. *Intrinsic Equation of the Curve*

By eliminating T between (18) and (15), we find

$$\rho = \frac{ds}{d\phi} = \frac{c}{\gamma}\sec^2\phi\,(\sec\phi + \tan\phi)^{a/g}. \quad (23)$$

This result for the radius of curvature agrees with that obtained by another method by Appell (*Acta Math.*, XII, 1889, p. 16).

The integration of (23) gives

$$s = \frac{c}{\gamma\left(1 - \frac{a^2}{g^2}\right)}(\sec\phi + \tan\phi)^{a/g}\left(\tan\phi - \frac{a}{g}\sec\phi\right)\Bigg]_{\phi_0}^{\phi}.$$

On substituting the value of c and putting

$$\frac{\sec\phi + \tan\phi}{\sec\phi_0 + \tan\phi_0} = \Phi,$$

we get

$$s = \frac{\frac{T_0}{\gamma} - \frac{v^2}{g}}{1 - \frac{a^2}{g^2}}\left[\Phi^{a/g}\frac{\tan\phi - \frac{a}{g}\sec\phi}{\tan\phi_0 - \frac{a}{g}\sec\phi_0} - 1\right]\left(\sin\phi_0 - \frac{a}{g}\right). \quad (24)$$

* This has been pointed out by Routh, Advanced Rigid Dynamics, Art. 524; and by Love, Theoretical Mechanics, Art. 270.

If (24) is expressed as a power series in $\phi - \phi_0$, we have

$$s = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right) \sec \phi_0 \left\{ \begin{aligned} &(\phi - \phi_0) + \sec \phi_0 \left(2 \sin \phi_0 + \frac{a}{g}\right)(\phi - \phi_0)^2/2! \\ &+ \sec^2 \phi_0 \left(4 \sin^2 \phi_0 + 5\frac{a}{g} \sin \phi_0 + 2 + \frac{a^2}{g^2}\right)(\phi - \phi_0)^3/3! \\ &+ \sec^3 \phi_0 \left(8 \sin^3 \phi_0 + 19\frac{a}{g} \sin^2 \phi_0 + 16 \sin \phi_0 \right. \\ &\left. + 9\frac{a^2}{g^2} \sin \phi_0 + 7\frac{a}{g} + \frac{a^3}{g^3}\right)(\phi - \phi_0)^4/4! + \cdots \end{aligned} \right\} \tag{25}$$

This series reduces to the series in (6) if we put $a = 0$ and $v = 0$

If we put $a = 0$ in (24), we get

$$s = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right) \cos \phi_0 (\tan \phi - \tan \phi_0).$$

This is in the same form as equation (4), the parameter being $T_0 - \gamma v^2/g$ instead of T_0. This has already been referred to in Art. 4. The arc s could be expressed as a power series in a/g, but ρ can be put in that form more simply by combining equations (15) and (21). This gives

$$\rho = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right) \frac{\sec^2 \phi}{\sec \phi_0} \left[1 + \sum_{n=1}^{\infty} \frac{\left(\frac{a}{g}\right)^n}{n!} \log_e{}^n \Phi\right]. \tag{26}$$

6. *Equation in Rectangular Coördinates*

The forces shown in Fig. 2 can be resolved into their horizontal and vertical components. On summing these components, we have

$$\Sigma X = d(T \cos \phi) = \frac{\gamma ds}{g} \frac{d^2x}{dt^2}, \tag{27}$$

$$\Sigma Y = d(T \sin \phi) - \gamma ds = \frac{\gamma ds}{g} \frac{d^2y}{dt^2}. \tag{28}$$

Let v_x and v_y be the corresponding velocities in the direction of the coördinate axes, then

$$v_x = v \cos \phi \qquad \text{and} \qquad v_y = v \sin \phi.$$

Therefore

$$\frac{d^2x}{dt^2} = \cos \phi \frac{dv}{dt} - v \sin \phi \frac{d\phi}{dt} = a \cos \phi - v \sin \phi \frac{d\phi}{dt}.$$

But $v = ds/dt$, $\sin \phi = dy/ds$ and $\cos \phi = dx/ds$. Therefore

$$\frac{d^2x}{dt^2} = a \frac{dx}{ds} - v^2 \sin \phi \frac{d\phi}{ds}.$$

Similarly,

$$\frac{d^2y}{dt^2} = a\frac{dy}{ds} + v^2 \cos\phi \frac{d\phi}{ds}.$$

Equations (27) and (28) can then be written

$$\frac{d}{ds}(T\cos\phi) + \frac{\gamma v^2}{g}\sin\phi\frac{d\phi}{ds} = \frac{\gamma}{g}a\frac{dx}{ds}, \tag{29}$$

$$\frac{d}{ds}(T\sin\phi) - \frac{\gamma v^2}{g}\cos\phi\frac{d\phi}{ds} = \frac{\gamma}{g}a\frac{dy}{ds} + \gamma. \tag{30}$$

Equation (29) could have been obtained by multiplying (12) and (13) by $\cos\phi$ and $\sin\phi$, respectively, and taking the difference. Similarly (30) could be found by multiplying (12) and (13) by $\sin\phi$ and $\cos\phi$, respectively, and taking their sum.

On integrating (29) and (30), we have

$$\left(T - \frac{\gamma v^2}{g}\right)\cos\phi = \frac{\gamma a x}{g} + C_1, \tag{31}$$

$$\left(T - \frac{\gamma v^2}{g}\right)\sin\phi = \gamma s + \frac{\gamma a y}{g} + C_2. \tag{32}$$

When $x = 0$ and $y = 0$, then $s = 0$, $T = T_0$, $\phi = \phi_0$. Therefore

$$C_1 = \left(T_0 - \frac{\gamma v^2}{g}\right)\cos\phi_0, \tag{33}$$

$$C_2 = \left(T_0 - \frac{\gamma v^2}{g}\right)\sin\phi_0. \tag{34}$$

On dividing (32) by (31), we find

$$\tan\phi = \frac{C_2 + \frac{\gamma a y}{g} + \gamma s}{C_1 + \frac{\gamma a x}{g}}. \tag{35}$$

Now, put $p = \tan\phi$ and solve for s. This gives

$$s = \left(\frac{C_1}{\gamma} + \frac{a}{g}x\right)p - \frac{a}{g}y - \frac{C_2}{\gamma}. \tag{36}$$

Therefore

$$ds = \left(\frac{C_1}{\gamma} + \frac{a}{g}x\right)dp + p\frac{a}{g}dx - \frac{a}{g}dy = \left(\frac{C_1}{\gamma} + \frac{a}{g}x\right)dp. \tag{37}$$

But $ds = \sqrt{1 + p^2}\,dx$. Therefore

$$\int_{\phi_0}^{\phi} \frac{dp}{(1 + p^2)^{1/2}} = \int_0^x \frac{dx}{\frac{C_1}{\gamma} + \frac{a}{g}x}. \tag{38}$$

By integrating (38), we get

$$\log_e (p + \sqrt{1 + p^2}) \Big]_{\phi_0}^{\phi} = \frac{g}{a} \log_e \left(\frac{C_1}{\gamma} + \frac{a}{g} x \right) \Big]_0^x .$$

Therefore

$$p + \sqrt{1 + p^2} = (\sec \phi_0 + \tan \phi_0) \left(1 + \frac{\gamma a x}{C_1 g} \right)^{g/a}. \tag{39}$$

For brevity put $\sec \phi_0 + \tan \phi_0 = A, \dfrac{C_1 g}{\gamma a} = k$. On solving for p, we get

$$p = \frac{1}{2} \left[A \left(1 + \frac{x}{k} \right)^{g/a} - \frac{1}{A} \left(1 + \frac{x}{k} \right)^{-g/a} \right]. \tag{40}$$

On integrating (40), we obtain

$$y = \frac{k}{2} \left[\frac{A \left\{ \left(1 + \frac{x}{k} \right)^{1+g/a} - 1 \right\}}{1 + \frac{g}{a}} - \frac{\left(1 + \frac{x}{k} \right)^{1-g/a} - 1}{A \left(1 - \frac{g}{a} \right)} \right], \qquad (a \neq \pm g), \tag{41}$$

as the equation of the curve in rectangular coördinates. This equation contains the constants a, g, k and A, that is, a, g, ϕ_0, v, γ and T_0. The velocity v occurs only in the combination $T_0 - \gamma v^2/g$, which may be regarded as a single parameter.

The special cases $a = \pm g$ are taken up in Art. 8.

If the origin is taken at the lowest point of the curve, $\phi_0 = 0$, $A = 1$ and

$$k = \frac{g}{a} \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right).$$

Then (41) becomes

$$y = \frac{g}{2a} \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right) \left[\frac{\left(1 + \dfrac{\frac{a}{g} x}{\frac{T_0}{\gamma} - \frac{v^2}{g}} \right)^{1+g/a} - 1}{1 + \frac{g}{a}} - \frac{\left(1 + \dfrac{\frac{a}{g} x}{\frac{T_0}{\gamma} - \frac{v^2}{g}} \right)^{1-g/a} - 1}{1 - \frac{g}{a}} \right]. \tag{42}$$

Equation (41) can be expanded into a power series in x, giving

$$y = x \tan \phi_0 + \frac{x^2 \sec^2 \phi_0}{2! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)} + \sum_{n=3}^{\infty} \frac{x^n \left(\frac{a}{g} \right)^{n-1}}{2 \cdot n! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^{n-1}} \left[A \prod_{h=0}^{n-2} \left(\frac{g}{a} - h \right) + (-1)^{n-2} \frac{1}{A} \prod_{h=0}^{n-2} \left(\frac{g}{a} + h \right) \right]. \tag{43}$$

For example, the coefficient of x^3 in (43) is

$$\frac{\sec^3 \phi_0 \left(\sin \phi_0 - \frac{a}{g} \right)}{3! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^2};$$

the coefficient of x^4 is

$$\frac{\sec^4 \phi_0 \left(1 - 3 \frac{a}{g} \sin \phi_0 + 2 \frac{a^2}{g^2} \right)}{4! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^3}.$$

If the origin is taken at the lowest point of the curve, (43) reduces to

$$y = \frac{x^2}{2! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)} - \frac{x^3 \frac{a}{g}}{3! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^2} + \frac{x^4 \left(1 + 2 \frac{a^2}{g^2} \right)}{4! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^3} - \frac{x^5 6 \frac{a}{g} \left(1 + \frac{a^2}{g^2} \right)}{5! \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right)^4} + \cdots. \tag{44}$$

The presence of the odd powers of x shows that the curve is not symmetrical with respect to the y-axis, and it is evident that $x = -x_1$ gives a larger ordinate than $x = x_1$.

7. *Parametric Representation*

If equation (39) is solved for x, we have

$$x = k \left[\left(\frac{\sec \phi + \tan \phi}{\sec \phi_0 + \tan \phi_0} \right)^{a/g} - 1 \right] = k[\Phi^{a/g} - 1]. \tag{45}$$

When expressed as a power series in a/g, it is

$$x = \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right) \cos \phi_0 \sum_{n=0}^{\infty} \frac{\log_e^{n+1} \Phi}{(n + 1)!} \left(\frac{a}{g} \right)^n. \tag{46}$$

When the expression for x in (45) is substituted in (41), we get

$$y = \frac{k}{2} \left[\frac{A \{ \Phi^{(a/g)+1} - 1 \}}{1 + \frac{g}{a}} - \frac{\Phi^{(a/g)-1} - 1}{A \left(1 - \frac{g}{a} \right)} \right]. \tag{47}$$

Equations (45) and (47) give us a parametric representation in terms of the parameter ϕ.

Equation (47) can be expanded into a power series in a/g, giving

$$y = \left(\frac{T_0}{\gamma} - \frac{v^2}{g} \right) \cos \phi_0 \sum_0^{\infty} A_k \left(\frac{a}{g} \right)^k, \tag{48}$$

in which

$$A_0 = \sec\phi - \sec\phi_0,$$

$$A_1 = \log_e\Phi\sec\phi - \tan\phi + \tan\phi_0,$$

$$A_2 = \log_e^2\Phi\left(\frac{\sec\phi}{2!}\right) - \log_e\Phi\tan\phi + \sec\phi - \sec\phi_0,$$

$$A_3 = \log_e^3\Phi\left(\frac{\sec\phi}{3!}\right) - \log_e^2\Phi\left(\frac{\tan\phi}{2!}\right) + \log_e\Phi\sec\phi - \tan\phi + \tan\phi_0.$$

In general,

$$A_n = A_{n-2} + \frac{\sec\phi\log_e^{n-1}\Phi}{(n-1)!}\left(\frac{\log_e\Phi}{n} - \sin\phi\right). \qquad (n \geqq 2).$$

If $\phi_0 = 0$, equations (45) and (47) reduce to

$$x = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right)\frac{g}{a}[(\sec\phi + \tan\phi)^{a/g} - 1], \tag{49}$$

$$y = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right)\frac{1}{2}\left[\frac{(\sec\phi + \tan\phi)^{(a/g)+1} - 1}{\frac{a}{g}+1} - \frac{(\sec\phi + \tan\phi)^{(a/g)-1} - 1}{\frac{a}{g}-1}\right]. \tag{50}$$

If the cable is to pass over pulleys at the two fixed points, O and P, the shape of the curve for any given value of a is independent of the velocity v. This can be shown as follows: Let the coördinates of P be x_1 and y_1, and let the inclination at P be denoted by ϕ_1. Equations (49) and (50) must be satisfied by (x_1, y_1). Therefore

$$x_1 = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right)\frac{g}{a}[(\sec\phi_1 + \tan\phi_1)^{a/g} - 1], \tag{51}$$

$$y_1 = \left(\frac{T_0}{\gamma} - \frac{v^2}{g}\right)\frac{1}{2}\left[\frac{(\sec\phi_1 + \tan\phi_1)^{(a/g)+1} - 1}{\frac{a}{g}+1} - \frac{(\sec\phi_1 + \tan\phi_1)^{(a/g)-1} - 1}{\frac{a}{g}-1}\right]. \tag{52}$$

The parameter $T_0/\gamma - v^2/g$ may be eliminated from these two equations, giving the relation

$$2y_1[(\sec\phi_1 + \tan\phi_1)^{a/g} - 1] = \frac{a}{g}x_1\left[\frac{(\sec\phi_1 + \tan\phi_1)^{(a/g)+1} - 1}{\frac{a}{g}+1} - \frac{(\sec\phi_1 + \tan\phi_1)^{(a/g)-1} - 1}{\frac{a}{g}-1}\right]. \tag{53}$$

This equation determines the angle ϕ_1. Therefore from (51),

$$\frac{T_0}{\gamma} - \frac{v^2}{g} = \frac{ax_1}{g[(\sec\phi_1 + \tan\phi_1)^{a/g} - 1]} = \text{Constant}. \tag{54}$$

If this value of $T_0/\gamma - v^2/g$ is substituted in (51) and (52), we will have a parametric representation independent of the velocity. That is, the shape of the curve depends on the acceleration a, but not on the velocity v.

8. *Special Cases*: $a = 0$; $a = \pm g$; $a = \infty$

If $a = 0$, equation (41) becomes indeterminate. But by evaluating the indeterminate forms involved, the equation reduces to that of the common catenary.

If $a = g$, equation (40) becomes

$$p = \frac{1}{2}\left[A\left(1 + \frac{x}{k}\right) - \frac{1}{A}\left(1 + \frac{x}{k}\right)^{-1}\right], \qquad \left(k = \frac{C_1}{\gamma}\right). \tag{55}$$

By integrating between proper limits, it is found that

$$y = \frac{1}{2}\frac{C_1}{\gamma}\left[\frac{A}{2}\left\{\left(1 + \frac{\gamma x}{C_1}\right)^2 - 1\right\} - \frac{1}{A}\log_e\left(1 + \frac{\gamma x}{C_1}\right)\right]. \tag{56}$$

The line $x = -C_1/\gamma$ is an asymptote to this curve.

Similarly, for $a = -g$, we have

$$y = \frac{1}{2}\frac{C_1}{\gamma}\left[\frac{\left(1 - \frac{\gamma x}{C_1}\right)^2 - 1}{2A} - A\log_e\left(1 - \frac{\gamma x}{C_1}\right)\right]. \tag{57}$$

This curve has the line $x = C_1/\gamma$ as an asymptote.

If $a = \infty$, equation (40) reduces to $p = \tan\phi_0$ by evaluating the indeterminate forms. Therefore

$$y = x\tan\phi_0,$$

the equation of a straight line through the origin.

PART II. EQUATIONS OF MOTION OF A FLEXIBLE, EXTENSIBLE CABLE IN A VERTICAL PLANE

This part will be taken up under three subdivisions: (*a*) the cable at rest; (*b*) moving with a velocity which corresponds to uniform velocity in the case of the inelastic cable; (*c*) moving with an acceleration which corresponds to constant acceleration in the case of the inelastic cable.

A. THE ELASTIC CATENARY

1. *The Intrinsic Equation of the Curve*

We use the term "elastic catenary" to designate the curve assumed by a flexible, extensible cord at rest. Let $d\sigma$ = length of unstretched element of the cable and let ds = length of the stretched element, then $ds = (1 + T\mu)d\sigma$, where $\mu = 1/b\lambda$, b = cross-section of the cable and λ = the modulus of elasticity of the material of the cable.

In Fig. 3 let the length of the cable from O to P equal s. The length of OP before stretching was σ. The weight therefore is $\gamma\sigma$. Taking the sum of the horizontal com-

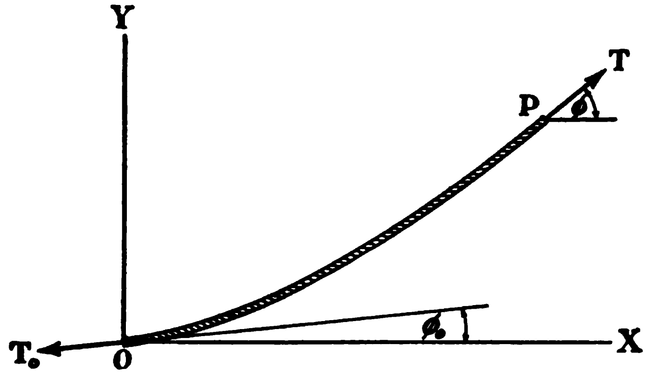


FIG.

ponents and the sum of the vertical components, we have the two equations:

$$\Sigma X = T\cos\phi - T_0\cos\phi_0 = 0, \tag{1}$$

$$\Sigma Y = T\sin\phi - T_0\sin\phi_0 - \gamma\sigma = 0. \tag{2}$$

From (1), we have

$$T = T_0\cos\phi_0\sec\phi. \tag{3}$$

On eliminating T from equations (1) and (2), we get

$$\tan\phi = \frac{\gamma\sigma}{T_0\cos\phi_0} + \tan\phi_0. \tag{4}$$

By differentiating equation (4) with respect to s, it is found that

$$\sec^2\phi\frac{d\phi}{ds} = \frac{\gamma\sec\phi_0}{T_0}\frac{d\sigma}{ds}. \tag{5}$$

But $\rho = ds/d\phi$, $ds/d\sigma = 1 + T\mu$ and $T = T_0\cos\phi_0\sec\phi$. On making these substitutions in (5), we obtain

$$\rho = \frac{T_0}{\gamma}\cos\phi_0(1 + T_0\mu\cos\phi_0\sec\phi)\sec^2\phi. \tag{6}$$

On integrating equation (6),

$$s = \frac{T_0}{\gamma}\cos\phi_0\left[\tan\phi + \frac{T_0\mu\cos\phi_0}{2}\{\tan\phi\sec\phi + \log_e(\sec\phi + \tan\phi)\}\right]_{\phi_0}^{\phi}. \tag{7}$$

Therefore

$$s = \frac{T_0}{\gamma}\cos\phi_0\left[\tan\phi - \tan\phi_0 + \frac{T_0\mu\cos\phi_0}{2}\{\tan\phi\sec\phi - \tan\phi_0\sec\phi_0 + \log_e\Phi\}\right], \tag{8}$$

where

$$\Phi = \frac{\sec\phi + \tan\phi}{\sec\phi_0 + \tan\phi_0}.$$

This is the intrinsic equation of the curve.

If s_c equals the length of the corresponding inelastic catenary, it follows from equation (8) that

$$s = s_c + \frac{T_0^2\mu}{2\gamma}\cos^2\phi_0\,\{\tan\phi\sec\phi - \tan\phi_0\sec\phi_0 + \log_e\Phi\}. \tag{9}$$

If the origin is at the lowest point of the curve, $\phi_0 = 0$, and equation (8) becomes

$$s = \frac{T_0}{\gamma}\left[\tan\phi + \frac{T_0\mu}{2}\{\tan\phi\sec\phi + \log_e(\sec\phi + \tan\phi)\}\right]. \tag{10}$$

Equation (8), when expanded into a power series in $\phi - \phi_0$, becomes

$$s = \frac{T_0}{\gamma}\sec\phi_0\left\{\begin{array}{l}(1 + T_0\mu)(\phi - \phi_0) + (2 + 3T_0\mu)\tan\phi_0\,(\phi - \phi_0)^2/2! \\ \quad + \{(4 + 9T_0\mu)\tan^2\phi_0 + (2 + 3T_0\mu)\sec^2\phi_0\}(\phi - \phi_0)^3/3! \\ \quad + \tan\phi_0\{(8 + 27T_0\mu)\tan^2\phi_0 + (16 + 33T_0\mu)\sec^2\phi_0\}(\phi - \phi_0)^4/4! \\ \quad + \{(16 + 81T_0\mu)\tan^4\phi_0 + (88 + 246T_0\mu)\sec^2\phi_0\tan^2\phi_0 \\ \quad + (16 + 33T_0\mu)\sec^4\phi_0\}(\phi - \phi_0)^5/5! + \cdots\end{array}\right\}. \tag{11}$$

2. *Parametric Representation*

On combining the relation $dx = \cos\phi\,ds$ with equation (6), we get

$$\frac{dx}{d\phi} = \frac{T_0}{\gamma}\cos\phi_0\sec\phi\,(1 + T_0\mu\cos\phi_0\sec\phi).$$

By integrating between proper limits, we obtain

$$x = \frac{T_0}{\gamma}\cos\phi_0\log_e\Phi + \frac{T_0^2\mu}{\gamma}\cos^2\phi_0\,(\tan\phi - \tan\phi_0). \tag{12}$$

In a similar manner, we obtain

$$y = \frac{T_0}{\gamma}\cos\phi_0\,(\sec\phi - \sec\phi_0) + \frac{T_0^2\mu}{2\gamma}\cos^2\phi_0\,(\sec^2\phi - \sec^2\phi_0). \tag{13}$$

These two equations furnish us a parametric representation of the curve. On comparing with the corresponding equations for the inelastic catenary, we see that

$$x = x_c + \mu\frac{T_0^2}{\gamma}\cos^2\phi_0\,(\tan\phi - \tan\phi_0), \tag{14}$$

$$y = y_c + \mu\frac{T_0^2}{2\gamma}\cos^2\phi_0\,(\sec^2\phi - \sec^2\phi_0), \tag{15}$$

where x_c and y_c are the coördinates of a point on the common catenary having the same inclination ϕ. From (14) and (15), it is evident that if $\phi > \phi_0$, then $x > x_c$ and $y > y_c$.

If $\phi_0 = 0$, we have

$$x = \frac{T_0}{\gamma}(gd^{-1}\phi + T_0\mu \cdot \tan\phi), \tag{16}$$

$$y = \frac{T_0}{\gamma}(\sec\phi - 1 + \tfrac{1}{2}T_0\mu \tan^2\phi). \tag{17}$$

3. *Equation in Rectangular Coördinates*

A relation between x and y may be found by eliminating ϕ from equations (12) and (13). When we solve (13) for $\sec\phi$, we get

$$\begin{aligned}\sec\phi &= \frac{-1 + \sqrt{1 + 2\gamma\mu\left\{y + \dfrac{T_0}{\gamma}\left(1 + \dfrac{T_0\mu}{2}\right)\right\}}}{T_0\mu\cos\phi_0}\\ &= \frac{2\gamma\left\{y + \dfrac{T_0}{\gamma}\left(1 + \dfrac{T_0\mu}{2}\right)\right\}}{T_0\cos\phi_0\left(1 + \sqrt{1 + 2\gamma\mu\left\{y + \dfrac{T_0}{\gamma}\left(1 + \dfrac{T_0\mu}{2}\right)\right\}}\right)}.\end{aligned} \tag{18}$$

Only the positive square root is used, as $\sec\phi$ is positive. On substituting this value of $\sec\phi$ in equation (12) and putting

$$\frac{T_0}{\gamma}\left(1 + \frac{T_0\mu}{2}\right) = B,$$

we have

$$\begin{aligned}x = \frac{T_0}{\gamma}\cos\phi_0\log_e &\frac{2(y+B) + \sqrt{4(y+B)^2 - 2\dfrac{T_0^2}{\gamma^2}\cos^2\phi_0\{1 + \gamma\mu(y+B) + \sqrt{1 + 2\gamma\mu(y+B)}\}}}{\dfrac{T_0}{\gamma}\cos\phi_0(\sec\phi_0 + \tan\phi_0)\{1 + \sqrt{1 + 2\gamma\mu(y+B)}\}}\\ &+ \frac{T_0^2\mu}{2\gamma}\cos^2\phi_0\left[\frac{\sqrt{4(y+B)^2 - 2\dfrac{T_0^2}{\gamma^2}\cos^2\phi_0\{1 + \gamma\mu(y+B) + \sqrt{1 + 2\gamma\mu(y+B)}\}}}{\dfrac{T_0}{\gamma}\cos\phi_0\{1 + \sqrt{1 + 2\gamma\mu(y+B)}\}} - \tan\phi_0\right].\end{aligned} \tag{19}$$

If we put $\mu = 0$, equation (19) reduces to

$$x = \frac{T_0}{\gamma}\cos\phi_0\log_e\frac{\sec\phi_0\left(y + \dfrac{T_0}{\gamma}\right) + \sqrt{\left(y + \dfrac{T_0}{\gamma}\right)\sec^2\phi_0 - \dfrac{T_0^2}{\gamma^2}}}{\sec\phi_0 + \tan\phi_0} \tag{20}$$

which is the same as equation (9) of Part I.

It is evident from the way that y is involved in equation (19) that it cannot be found as a function of x except as a power series. To obtain this power series in x, we proceed as follows:

Put $p = \tan\phi$. $dp/dx = \sec^2\phi\, d\phi/dx = \sec^2\phi\, d\phi/ds\, ds/dx$. But $\rho = ds/d\phi$ and $\sec\phi = ds/dx$. On making these substitutions, we get

$$\frac{dp}{dx} = \frac{\sec^3\phi}{\rho}. \tag{22}$$

Since ϕ and ρ are both functions of x, the second member of (22) is a function of x. Let $F(x)$ denote this function. Therefore

$$\frac{dp}{dx} = F(x) = F(0) + F'(0)x + F''(0)\frac{x^2}{2!} + F'''(0)\frac{x^3}{3!} + F^{\text{iv}}(0)\frac{x^4}{4!} + \cdots. \tag{23}$$

On integrating, we obtain

$$p = \frac{dy}{dx} = \tan\phi_0 + F(0)x + F'(0)\frac{x^2}{2!} + F''(0)\frac{x^3}{3!} + F'''(0)\frac{x^4}{4!} + \cdots. \tag{24}$$

A second integration gives (since $x = 0$ when $y = 0$),

$$y = x\tan\phi_0 + F(0)\frac{x^2}{2!} + F'(0)\frac{x^3}{3!} + F''(0)\frac{x^4}{4!} + F'''(0)\frac{x^5}{5!} + \cdots. \tag{25}$$

In getting the successive derivatives we will need dT/dx. But dT/dx equals $\sec\phi\, dT/ds$. On differentiating equations (1) and (2), we get

$$\cos\phi\frac{dT}{ds} - T\sin\phi\frac{d\phi}{ds} = 0,$$

$$\sin\phi\frac{dT}{ds} + T\cos\phi\frac{d\phi}{ds} = \gamma\frac{d\sigma}{ds} = \frac{\gamma}{1 + T\mu}.$$

Therefore

$$\frac{dT}{ds} = \frac{\gamma\sin\phi}{1 + T\mu},$$

whence

$$\frac{dT}{dx} = \frac{\gamma\tan\phi}{1 + T\mu}.$$

Since

$$F(x) = \frac{\sec^3\phi}{\rho},$$

it follows that

$$F(0) = \frac{\sec^3\phi_0}{\rho_0},$$

where ρ_0 is the value of ρ at the origin. By successive differentiation and putting $x = 0$, we get

$$F'(0) = \frac{\sec^5\phi_0\sin\phi_0}{\rho_0^2(1 + T_0\mu)}; \qquad F''(0) = \frac{\sec^7\phi_0}{\rho_0^3(1 + T_0\mu)}\left[1 - \frac{3T_0\mu\sin^2\phi_0}{1 + T_0\mu}\right];$$

$$F'''(0) = \frac{\sec^9\phi_0\sin\phi_0}{\rho_0^4(1 + T_0\mu)}\left[1 - \frac{10T_0\mu}{1 + T_0\mu} + \frac{15T_0^2\mu^2\sin^2\phi_0}{(1 + T_0\mu)^2}\right].$$

These values substituted in (25) gives an expansion of y in powers of x.

If we put $\mu = 0$ in the foregoing series we obtain the catenary series in equation (11) of Part I. The catenary series is known to be convergent for all values of x. The five terms shown in (25) can be shown to be equal to or less than the corresponding terms in the catenary series. This makes it probable that the series in (25) is also convergent.

B. THE CABLE MOVING WITH A VELOCITY CORRESPONDING TO UNIFORM VELOCITY OF THE INELASTIC CABLE

4. *The Intrinsic Equation of the Curve*

Suppose the elastic cable to be moving in such a way that its velocity would reduce to a constant if its coefficient of elasticity were infinite. Since $ds = (1 + T\mu)d\sigma$, it follows that $v' = (1 + T\mu)v$, where v is the velocity of the inelastic cable and v' is the velocity of the elastic cable. The acceleration along the elastic cable is $a' = dv'/dt$, while $dv/dt = 0$. Therefore, $a' = \mu v dT/dt$.

On considering the forces acting on an element ds of the stretched (equals $d\sigma$ of the unstretched) cable, we have

$$\Sigma T = dT - \gamma \sin \phi d\sigma = a'\gamma \frac{d\sigma}{g} = \mu v \frac{dT}{dt} \gamma \frac{d\sigma}{g}.$$

On using the relations $v = d\sigma/dt$ and $ds = (1 + T\mu)d\sigma$, this equation becomes

$$\Sigma T = \left(1 - \gamma\mu\frac{v^2}{g}\right) dT = \frac{\gamma \sin \phi ds}{1 + T\mu}. \tag{26}$$

Therefore

$$(1 + T\mu)\left(1 - \gamma\mu\frac{v^2}{g}\right)\frac{dT}{ds} = \gamma \sin \phi. \tag{27}$$

In a similar way, we get

$$\Sigma N = Td\phi - \gamma \cos \phi d\sigma = \frac{v'^2}{\rho}\gamma\frac{d\sigma}{g} = (1 + T\mu)\gamma v^2 \frac{d\phi}{g}. \tag{28}$$

Equation (28) reduces to

$$(1 + T\mu)\left[T - (1 + T\mu)\gamma\frac{v^2}{g}\right]\frac{d\phi}{ds} = \gamma \cos \phi. \tag{29}$$

On dividing (27) by (29), we get

$$\frac{1 - \gamma\mu\dfrac{v^2}{g}}{T\left(1 - \gamma\mu\dfrac{v^2}{g}\right) - \gamma\dfrac{v^2}{g}} dT = \tan \phi \, d\phi. \tag{30}$$

By integration, we have

$$\log_e \left[T\left(1 - \gamma\mu\frac{v^2}{g}\right) - \gamma\frac{v^2}{g}\right] = \log_e \sec \phi + \log_e C. \tag{31}$$

Therefore

$$T\left(1 - \gamma\mu\frac{v^2}{g}\right) - \gamma\frac{v^2}{g} = C \sec \phi. \tag{32}$$

Since $T = T_0$ when $\phi = \phi_0$, it follows that

$$C = \cos\phi_0 \left[T_0 \left(1 - \gamma\mu \frac{v^2}{g} \right) - \gamma \frac{v^2}{g} \right]. \tag{33}$$

On solving (32) for T and substituting in (29), we have

$$\frac{ds}{d\phi} = \rho = \frac{C(1 + C\mu \sec\phi) \sec^2\phi}{\gamma\left(1 - \gamma\mu \frac{v^2}{g}\right)}. \tag{34}$$

From the two-fold manner in which v enters equation (34), through the factor $1 - \gamma\mu v^2/g$ and the constant C, it is evident that the curve is different from the elastic catenary at rest given by equation (6).

By integration of (34), we get

$$s = \frac{C}{\gamma\left(1 - \gamma\mu \frac{v^2}{g}\right)} \left[\tan\phi + \frac{C\mu}{2} \{\tan\phi \sec\phi + \log_e (\sec\phi + \tan\phi)\} \right]_{\phi_0}^{\phi}. \tag{35}$$

Therefore

$$s = \frac{C}{\gamma\left(1 - \gamma\mu \frac{v^2}{g}\right)} \left[\tan\phi - \tan\phi_0 + \frac{C\mu}{2} (\tan\phi \sec\phi - \tan\phi_0 \sec\phi_0 + \log_e \Phi) \right]. \tag{36}$$

If we put $v = 0$ in equation (36), the equation reduces to (8), the intrinsic equation of the elastic catenary.

5. *Parametric Representation*

Since $dx = \cos\phi\, ds = \rho \cos\phi\, d\phi$ and $dy = \rho \sin\phi\, d\phi$, we have from equation (34)

$$dx = \frac{C(1 + C\mu \sec\phi) \sec\phi d\phi}{\gamma\left(1 - \gamma\mu \frac{v^2}{g}\right)},$$

$$dy = \frac{C(1 + C\mu \sec\phi) \tan\phi \sec\phi d\phi}{\gamma\left(1 - \gamma\mu \frac{v^2}{g}\right)}.$$

By integrating between proper limits, we get

$$x = \frac{C}{\gamma} \frac{\log_e \Phi + C\mu (\tan\phi - \tan\phi_0)}{1 - \gamma\mu \frac{v^2}{g}}, \tag{37}$$

$$y = \frac{C}{\gamma} \frac{\sec\phi - \sec\phi_0 + \frac{1}{2} C\mu (\sec\phi - \sec\phi_0)}{1 - \gamma\mu \frac{v^2}{g}}. \tag{38}$$

Equations (37) and (38) give a parametric representation in terms of the inclination ϕ.

6. *The Equation of the Curve in Rectangular Coördinates*

The parameter ϕ is easily eliminated between equations (37) and (38). The resulting equation is similar to that obtained for the elastic cable at rest in Article 3.

C. THE CABLE MOVING WITH AN ACCELERATION CORRESPONDING TO A UNIFORM ACCELERATION OF THE INELASTIC CABLE

7. *The Intrinsic Equation of the Curve*

As in B, we have the relation $v' = (1 + T\mu)v$, where v' and v are the velocities along the stretched and unstretched cables, respectively. Let a' and a represent the corresponding accelerations. Then, $a' = a(1 + T\mu) + \mu v dT/dt$.

On summing the tangential and normal components of the forces acting on an element ds of the stretched (or $d\sigma$ of the unstretched) cable, we have

$$\Sigma T = dT - \gamma \sin\phi d\sigma = a'\gamma\frac{d\sigma}{g} = \left(a(1 + T\mu)\frac{\gamma}{g} + \mu\gamma v^2\frac{dT}{g}\right)d\sigma, \tag{39}$$

$$\Sigma N = Td\phi - \gamma\cos\phi\, d\sigma = \frac{v'^2}{\rho}\gamma\frac{d\sigma}{g} = (1 + T\mu)^2\frac{v^2}{\rho}\gamma\frac{d\sigma}{g}. \tag{40}$$

These equations may be written respectively

$$\left(1 - \gamma\mu\frac{v^2}{g}\right)\frac{dT}{ds} = \gamma\left(\frac{\sin\phi}{1 + T\mu} + \frac{a}{g}\right), \tag{41}$$

$$\frac{T}{\rho} = \frac{\gamma\cos\phi}{1 + T\mu} + (1 + T\mu)\frac{v^2}{\rho}\frac{\gamma}{g}. \tag{42}$$

On solving (42) for ρ, we have

$$\rho = \sec\phi\,(1 + T\mu)\left[\frac{T}{\gamma} - (1 + T\mu)\frac{v^2}{g}\right]. \tag{43}$$

The right member of this equation can be expanded into a power series in $\phi - \phi_0$ by making use of (41) to eliminate $dT/d\phi$, since $dT/d\phi = \rho dT/ds$. On denoting the successive derivatives of ρ by $\rho', \rho'', \rho''', \cdots$ and their values at the origin by $\rho_0', \rho_0'', \rho_0''',$ $\cdots$, we have

$$\frac{ds}{d\phi} = \rho = \rho_0 + \rho_0'(\phi - \phi_0) + \frac{\rho_0''(\phi - \phi_0)^2}{2!} + \frac{\rho_0'''(\phi - \phi_0)^3}{3!} + \cdots. \tag{44}$$

The coefficients $\rho_0, \rho_0', \rho_0'', \rho_0'''$, are

$$\rho_0 = \sec\phi_0\,(1 + T_0\mu)\left[\frac{T_0}{\gamma}\left(1 - \gamma\mu\frac{v^2}{g}\right) - \frac{v^2}{g}\right],$$

$$\rho_0' = \rho_0\sec\phi_0\left[\sin\phi_0\left(2 + \frac{\mathrm{T}_0}{1 + T_0\mu}\right) + \frac{a}{g}(1 + T_0\mu + \mathrm{T}_0)\right],\ \left(\mathrm{T}_0 = T_0\mu - \frac{\gamma\mu v^2}{g - \gamma\mu v^2}\right),$$

$$\rho_0'' = \rho_0 \sec^2 \phi_0 \left[\begin{aligned} &\sin^2 \phi_0 \left(4 + \frac{5T_0}{1 + T_0\mu}\right) + \frac{a}{g} \sin \phi_0 \left\{ 5(1 + T_0\mu) + 10T_0 + \frac{T_0^2}{1 + T_0\mu} \right\} \\ &+ 2 + \frac{T_0}{1 + T_0\mu} + \frac{a^2}{g^2} \{(1 + T_0\mu)^2 + 4T_0(1 + T_0\mu) + T_0^2\} \end{aligned} \right],$$

$$\rho_0''' = \rho_0 \sec^3 \phi_0 \left[\begin{aligned} &\sin^3 \phi_0 \left(8 + \frac{19T_0}{1 + T_0\mu}\right) + \frac{a}{g} \sin^2 \phi_0 \left\{ 19(1 + T_0\mu) + 64T_0 + 15\frac{T_0^2}{1 + T_0\mu} \right\} \\ &+ \sin \phi_0 \left(16 + \frac{17T_0}{1 + T_0\mu}\right) \\ &+ \frac{a^2}{g^2} \sin \phi_0 \left\{ 9(1 + T_0\mu)^2 + 53(1 + T_0\mu)T_0 + 27T_0^2 + \frac{T_0^3}{1 + T_0\mu} \right\} \\ &+ \frac{a}{g} \left\{ 7(1 + T_0\mu) + 14T_0 + \frac{T_0^2}{1 + T_0\mu} \right\} \\ &+ \frac{a^3}{g^3} \{(1 + T_0\mu)^3 + 11(1 + T_0\mu)^2T_0 + 11(1 + T_0\mu)T_0^2 + T_0^3\} \end{aligned} \right].$$

On integrating the series in (44), we obtain

$$s = \rho_0(\phi - \phi_0) + \frac{\rho_0'(\phi - \phi_0)^2}{2!} + \frac{\rho_0''(\phi - \phi_0)^3}{3!} + \frac{\rho_0'''(\phi - \phi_0)^4}{4!} + \cdots. \tag{45}$$

The series represented in (45) reduces to (25) of Part I by putting $\mu = 0$. As a further check, if we put $v = 0$, $a = 0$, in (45), we get (11), the intrinsic equation of an extensible cable at rest.

8. *Parametric Representation*

We use the same method of derivation as in the other cases. Since $dx/d\phi = \rho \cos \phi$ and ρ is a function of ϕ, we let $dx/d\phi = F_1(\phi) = \rho \cos \phi$.

$$F_1'(\phi) = \rho' \cos \phi - \rho \sin \phi, \qquad F_1'(\phi_0) = \rho_0' \cos \phi_0 - \rho_0 \sin \phi_0.$$

But from Article 7 we have

$$\rho_0' = \rho_0 \sec \phi_0 \left\{ \sin \phi_0 \left(2 + \frac{T_0}{1 + T_0\mu}\right) + \frac{a}{g}(1 + T_0\mu + T_0) \right\}.$$

Therefore

$$F_1'(\phi_0) = \rho_0(1 + T_0\mu + T_0)\left(\frac{\sin \phi_0}{1 + T_0\mu} + \frac{a}{g}\right).$$

$$F_1''(\phi) = \rho'' \cos \phi - 2\rho' \sin \phi - \rho \cos \phi,$$

which leads to

$$F_1''(\phi_0) = \rho_0 \sec \phi_0 \left[\begin{aligned} &\sin^2 \phi_0 \left(1 + \frac{3T_0}{1 + T_0\mu}\right) + \frac{a}{g} \sin \phi_0 \left\{ 3(1 + T_0\mu) + 8T_0 + \frac{T_0^2}{1 + T_0\mu} \right\} \\ &+ \frac{a^2}{g^2} \{(1 + T_0\mu)^2 + 4(1 + T_0\mu)T_0 + T_0^2\} + 1 + \frac{T_0}{1 + T_0\mu} \end{aligned} \right].$$

In a similar manner, we get

$$F_1'''(\phi_0) = \rho_0^2 \sec^3\phi_0 \left[\begin{array}{l} \sin^3\phi_0\left(1+\dfrac{7\mathrm{T}_0}{1+T_0\mu}\right) \\ +\dfrac{a}{g}\sin^2\phi_0\left\{7(1+T_0\mu)+37\mathrm{T}_0+12\dfrac{\mathrm{T}_0^2}{1+T_0\mu}\right\} \\ +\sin\phi_0\left(5+11\dfrac{\mathrm{T}_0}{1+T_0\mu}\right) \\ +\dfrac{a^2}{g^2}\sin\phi_0\left\{6(1+T_0\mu)^2+41(1+T_0\mu)\mathrm{T}_0+24\mathrm{T}_0^2+\dfrac{\mathrm{T}_0^3}{1+T_0\mu}\right\} \\ +\dfrac{a}{g}\left\{4(1+T_0\mu)+11\mathrm{T}_0+\dfrac{\mathrm{T}_0^2}{1+T_0\mu}\right\} \\ +\dfrac{a^3}{g^3}\{(1+T_0\mu)^3+11(1+T_0\mu)^2\mathrm{T}_0+11(1+T_0\mu)\mathrm{T}_0^2+\mathrm{T}_0^3\} \end{array}\right].$$

On expanding $F_1(\phi)$ into a power series in $\phi - \phi_0$, we have

$$\begin{aligned} \frac{dx}{d\phi} = F_1(\phi) = F_1(\phi_0) + F_1'(\phi_0)(\phi-\phi_0) + F_1''(\phi_0)\frac{(\phi-\phi_0)^2}{2!} \\ + F_1'''(\phi_0)\frac{(\phi-\phi_0)^3}{3!} + \cdots. \end{aligned} \tag{46}$$

By integration, we get

$$\begin{aligned} x = F_1(\phi_0)(\phi-\phi_0) + F_1'(\phi_0)\frac{(\phi-\phi_0)^2}{2!} + F_1''(\phi_0)\frac{(\phi-\phi_0)^3}{3!} \\ + F_1'''(\phi_0)\frac{(\phi-\phi_0)^4}{4!} + \cdots. \end{aligned} \tag{47}$$

If we put $\mu = 0$ in (47), we get the expansion of (45) of Part I, and if we put $a = 0$ and $v = 0$ in (47), we have a series which is the expansion of (12).

We obtain a series for y in a similar way, calling the function $F_2(\phi)$. Therefore $dy/d\phi = F_2(\phi) = \rho\sin\phi$ and $F_2(\phi_0) = \rho_0 \sin\phi_0$.

$$F_2'(\phi) = \rho'\sin\phi + \rho\cos\phi, \text{ and } F_2'(\phi_0) = \rho_0'\sin\phi_0 + \rho_0\cos\phi_0,$$

$$F_2'(\phi_0) = \rho_0\sec\phi_0\left\{\sin\phi_0\left(1+\frac{\mathrm{T}_0}{1+T_0\mu}\right)+\frac{a}{g}\sin\phi_0(1+T_0\mu+\mathrm{T}_0)+1\right\},$$

$$F_2''(\phi) = \rho''\sin\phi + 2\rho'\cos\phi - \rho\sin\phi,$$

$$F_2''(\phi_0) = \rho_0\sec^2\phi_0\left[\begin{array}{l} \sin^3\phi_0\left(1+\dfrac{3\mathrm{T}_0}{1+T_0\mu}\right)+\dfrac{a}{g}\sin^2\phi_0\left\{3(1+T_0\mu)+8\mathrm{T}_0+\dfrac{\mathrm{T}_0^2}{1+T_0\mu}\right\} \\ +\sin\phi_0\left(5+\dfrac{3\mathrm{T}_0}{1+T_0\mu}\right) \\ +\dfrac{a^2}{g^2}\sin\phi_0\{(1+T_0\mu)^2+4(1+T_0\mu)\mathrm{T}_0+\mathrm{T}_0^2\}+\dfrac{2a}{g}(1+T_0\mu+\mathrm{T}_0) \end{array}\right],$$

$$F_2'''(\phi) = \rho''' \sin\phi + 3\rho'' \cos\phi - 3\rho' \sin\phi - \rho \cos\phi,$$

$$F_2'''(\phi_0) = \rho_0 \sec^3\phi_0 \left[\begin{aligned} &\sin^4\phi_0\left(1 + \frac{7T_0}{1 + T_0\mu}\right) + \frac{a}{g}\sin^3\phi_0\left\{7(1 + T_0\mu) + 37T_0\right. \\ &\qquad\left. + 12\frac{T_0^2}{1 + T_0\mu}\right\} + \sin^2\phi_0\left(18 + 26\frac{T_0}{1 + T_0\mu}\right) \\ &+ \frac{a^2}{g^2}\sin^2\phi_0\left\{6(1 + T_0\mu^2) + 41(1 + T_0\mu)T_0 + 24T_0^2 + \frac{T_0^3}{1 + T_0\mu}\right\} \\ &\qquad + \frac{a}{g}\sin\phi_0\left\{19(1 + T_0\mu) + 41T_0 + \frac{4T_0^2}{1 + T_0\mu}\right\} \\ &+ \frac{a^3}{g^3}\sin\phi_0\{(1 + T_0\mu)^3 + 11(1 + T_0\mu)^2T_0 + 11(1 + T_0\mu)T_0^2 + T_0^3\} \\ &\qquad + \frac{a^2}{g^2}\{3(1 + T_0\mu)^2 + 12(1 + T_0\mu)T_0 + T_0^2\} \\ &\qquad + 5 + \frac{3T_0}{1 + T_0\mu} \end{aligned}\right].$$

Since

$$\frac{dy}{d\phi} = F_2(\phi) = F_2(\phi_0) + F_2'(\phi_0)(\phi - \phi_0) + F_2''(\phi_0)\frac{(\phi - \phi_0)^2}{2!} + F_2'''(\phi_0)\frac{(\phi - \phi_0)^3}{3!} + \cdots, \tag{48}$$

therefore

$$y = F_2(\phi_0)(\phi - \phi_0) + F_2'(\phi_0)\frac{(\phi - \phi_0)^2}{2!} + F_2''(\phi_0)\frac{(\phi - \phi_0)^3}{3!} + F_2'''(\phi_0)\frac{(\phi - \phi_0)^4}{4!} + \cdots. \tag{49}$$

If $\mu = 0$, series (49) reduces to the expansion of (47) of Part I, and reduces to the expansion of (13) if $a = 0$ and $v = 0$.

9. *Equation of the Curve in Rectangular Coördinates*

The method of derivation is similar to that used in Article 3. Put $p = \tan\phi$. Therefore $dp/dx = \sec^2\phi\, d\phi/dx$. But $d\phi/dx = \sec\phi/\rho$. Therefore $dp/dx = \sec^3\phi/\rho$. Since ϕ and ρ both depend on x, it follows that dp/dx is a function of x, which we will denote by $F_3(x)$, and $F_3(0) = \sec^3\phi_0/\rho_0$.
Therefore

$$F_3'(x) = \frac{3\rho\sec^3\phi\tan\phi\dfrac{d\phi}{dx} - \sec^3\phi\dfrac{d\rho}{dx}}{\rho^2}.$$

But $d\phi/dx = \sec\phi/\rho$ and $d\rho/dx = d\rho/d\phi\; d\phi/dx = \rho'/\rho\sec\phi$. Therefore

$$F_3'(0) = \frac{\sec^5\phi_0}{\rho_0^2}\left[\sin\phi_0\left(1 - \frac{T_0}{1 + T_0\mu}\right) - \frac{a}{g}(1 + T_0\mu + T_0)\right].$$

Similarly,

$$F_3''(0) = \frac{\sec^7 \phi_0}{\rho_0^3}\left[\begin{array}{l} \sin^2 \phi_0 \left(\dfrac{T_0}{1 + T_0\mu} - 1\right)\dfrac{3T_0}{1 + T_0\mu} \\ \qquad + \dfrac{a}{g}\sin \phi_0 \left\{ \dfrac{5T_0^2}{1 + T_0\mu} - 2T_0 - 3(1 + T_0\mu) \right\} \\ \qquad + 2\dfrac{a^2}{g^2}\{(1 + T_0\mu)^2 + (1 + T_0\mu)T_0 + T_0^2\} + 1 - \dfrac{T_0}{1 + T_0\mu} \end{array}\right].$$

$$\frac{dp}{dx} = F_3(x) = F_3(0) + F_3'(0)x + F_3''(0)x^2/2! + F_3'''(0)x^3/3! + F_3^{iv}(0)x^4/4! + \cdots. \quad (50)$$

On integrating twice, remembering that $p = \tan \phi$ and $y = 0$ when $x = 0$, we get

$$y = x \tan \phi_0 + F_3(0)x^2/2! + F_3'(0)x^3/3! + F_3''(0)x^4/4! + F_3'''(0)x^5/5! + \cdots. \quad (51)$$

Series (51) reduces to (43) of Part I by putting $\mu = 0$, and reduces to (25) if we put $a = 0$ and $v = 0$.

PART III. GRAPHIC COMPARISON OF CURVES

For comparison we will let the common catenary, the elastic catenary and the accelerated common catenary all pass through two points P_1 and P_2 fulfilling approximately the conditions outlined in the introduction. The inclination at P_1 is taken as 34 degrees for the common catenary, the difference in abscissas being 3,730 feet. The loaded car weighs 30,000 pounds, and the attached cable is a steel cable 1.375 inches in diameter weighing three pounds per linear foot. The modulus of elasticity λ is taken as 30,000,000 pounds per square inch. In making the comparison, the inelastic and elastic cables are assumed to have the same weight between the points P_1 and P_2.

The velocity for the accelerated inelastic cable is taken as zero, that is, as the cable starts with a given acceleration.

A. THE COMMON CATENARY

In Part I we found

$$x = \frac{T_0}{\gamma} gd^{-1} \phi = \frac{T_0}{\gamma} \log_e (\sec \phi + \tan \phi),$$

$$y = \frac{T_0}{\gamma} (\sec \phi - 1).$$

If the cable curve is the same as the track curve when the car is at P_1, then the tension T_1 in the cable equals $W \sin \phi_1$, where W is the weight of the loaded car and ϕ_1 is the inclination at P_1. But $T_1 \cos \phi_1 = T_0$. Therefore

$$T_0 = W \sin \phi_1 \cos \phi_1, \qquad \frac{T_0}{\gamma} = \frac{1}{2}\frac{W}{\gamma} \sin 2\phi_1.$$

Since $W = 30{,}000$, $\gamma = 3$ and $\phi_1 = 34°$, we get $T_0/\gamma = 4{,}636$ pounds. This gives for the point $P_1(x_1, y_1)$ the coördinates $x_1 = 2928.4$, $y_1 = 956.0$. Since $x_2 - x_1 = 3{,}730$,

$x_2 = 6,658.4$. The ordinate y_2 is found to be 5,661.7, and the inclination ϕ_2 is 63° 14′ 38″. On assuming values of x differing by 500, the corresponding ordinates and inclinations are shown in Table I which follows. In computing these values the relation

$$y + \frac{T_0}{\gamma} = \frac{T_0}{\gamma} \cosh \frac{x\gamma}{T_0}$$

was used as a check. The length of the curve from P_1 to P_2 was found to be 6,068.43 feet. The weight of the cable between these points is then 18,205.29 pounds.

B. THE ELASTIC CATENARY

For the elastic catenary we found in equations (16) and (17), Part II,

$$x' = \frac{T_0}{\gamma}(gd^{-1}\,\phi' + T_0\mu \tan \phi'),$$

$$y' = \frac{T_0}{\gamma}(\sec \phi' - 1 + \tfrac{1}{2}T_0\mu \tan^2 \phi'),$$

where T_0 equals the tension at the origin, γ equals the weight of the cable per linear foot, and x' and y' are the coördinates of a point on the elastic catenary.

The quantity T_0/γ is to be determined so that the cable shall pass through the points P_1 and P_2, the weight of the cable being the same as before stretching. By equation (4), Part II, we have

$$\sigma = \frac{T_0}{\gamma} \tan \phi' \text{ (since } \phi_0' = 0).$$

Therefore

$$\sigma_2 - \sigma_1 = \frac{T_0}{\gamma}(\tan \phi_2' - \tan \phi_1') = 6,068.43.$$

We also have

$$x_2' - x_1' = 3,730, \qquad y_2' - y_1' = 4,705.7.$$

Therefore we have the following set of three equations to determine ϕ_1', ϕ_2' and T_0/γ:

$$3,730\frac{\gamma}{T_0} = gd^{-1}\,\phi_2' - gd^{-1}\,\phi_1' + T_0\mu(\tan \phi_2' - \tan \phi_1'),$$

$$4,705.7\frac{\gamma}{T_0} = \sec \phi_2' - \sec \phi_1' + \tfrac{1}{2}T_0\mu(\tan^2 \phi_2' - \tan^2 \phi_1'),$$

$$6,068.43\frac{\gamma}{T_0} = \tan \phi_2' - \tan \phi_1'.$$

By successive trials we find that these equations are satisfied by $\phi_2' = 63° 28' 45.5''$, $\phi_1' = 33° 27' 17.5''$ and $T_0/\gamma = 4,518.16$. This leads to

$$x_1' = 2,803.1, \qquad y_1' = 897.5, \qquad x_2' = 6,533.1, \qquad y_2' = 5,603.2.$$

Since the points (x_1', y_1') and (x_2', y_2') are the same as (x_1, y_1) and (x_2, y_2) respectively, it follows that the origin O' has the coördinates (125.3, 58.5) with respect to the coördinate axes of the common catenary, that is $x = x' + 125.3$, $y = y' + 58.5$. The relative position of the axes and the curves (not drawn to scale) are shown graphically in Fig. 4.

The following table gives the corresponding values for x and y and the inclinations at various points on the inelastic and elastic catenaries. The points chosen have a common abscissa. The last column shows the difference of the corresponding ordinates.

TABLE I

Common Catenary $\mu=0,\ a=0$			Elastic Catenary $\mu \neq 0,\ a=0,\ v=0$				Diff. of Ords.
x	y	ϕ	x'	y'	y	ϕ'	
0	0	00° 00′ 00″	−125.3	0.2	58.7	−1° 35′ 20″	−58.7
500	27.0	6° 10′ 3″	374.7	15.5	74.0	4° 44′ 42″	−47.0
1,000	108.3	12° 15′ 50″	874.7	85.1	143.6	11° 1′ 15″	−35.3
1,500	244.8	18° 13′ 32″	1,374.7	210.7	269.2	17° 9′ 55″	−24.4
2,000	438.2	23° 59′ 4″	1,874.7	394.2	452.7	23° 6′ 47″	−14.5
2,500	690.6	29° 29′ 50″	2,374.7	638.1	696.6	28° 48′ 13″	− 6.0
P_1 2,928.4	956.0	34° 00′ 00″	2,803.1	897.5	956.0	33° 27′ 17″.5	0
3,000	1,004.9	34° 43′ 49″	2,874.7	945.5	1,004.0	34° 12′ 32″	0.9
3,500	1,385.3	39° 39′ 7″	3,374.7	1,319.6	1,378.1	39° 17′ 2″.5	7.2
4,000	1,835.3	44° 14′ 33″	3,874.7	1,765.2	1,823.7	44° 00′ 59″	11.6
4,500	2,361.1	48° 30′ 17″	4,374.7	2,285.6	2,344.1	48° 23′ 57″	17.0
5,000	2,967.8	52° 25′ 59″	4,874.7	2,893.8	2,952.3	52° 26′ 3″	15.5
5,500	3,667.7	56° 2′ 36″	5,374.7	3,590.6	3,649.1	56° 7′ 50″	14.6
6,000	4,455.6	59° 20′ 30″	5,874.7	4,386.7	4,445.2	59° 30′ 11″	10.4
6,500	5,353.2	62° 20′ 53″	6,374.7	5,291.9	5,350.4	62° 34′ 8″	2.8
P_2 6,658.4	5,661.7	63° 14′ 38″	6,533.1	5,603.2	5,661.7	63° 28′ 45″.5	0
7,000	6,368.0	65° 5′ 00″	6,874.7	6,317.3	6,375.8	65° 20′ 53″	− 7.8

C. THE ACCELERATED INELASTIC CATENARY

We found in this case that

$$x'' = \frac{T_0}{\gamma}\frac{g}{a}\left[(\sec\phi'' + \tan\phi'')^{a/g} - 1\right],$$

$$y'' = \frac{1}{2}\frac{T_0}{\gamma}\left[\frac{(\sec\phi'' + \tan\phi'')^{(a/g)+1} - 1}{\dfrac{a}{g} + 1} - \frac{(\sec\phi'' + \tan\phi'')^{(a/g)-1} - 1}{\dfrac{a}{g} - 1}\right].$$

As in A and B, T_0/γ is to be determined, so that the cable shall pass through the points P_1 and P_2. Therefore $x_2'' - x_1'' = 3{,}730$, $y_2'' - y_1'' = 4{,}705.7$. As in case A we assume the car to be at P_1 and that the cable curve is the same as the track curve, that is, that the load is just balanced by the tension in the cable. Therefore we have by equation (20), Part I,

$$W \sin\phi_1'' = T_0 \sec\phi_1'' (\sec\phi_1'' + \tan\phi_1'')^{a/g},$$

$$\frac{T_0}{\gamma} = \frac{W \sin 2\phi_1''}{2\gamma\,(\sec\phi_1'' + \tan\phi'')^{a/g}}.$$

Let the initial velocity be zero and let the acceleration a be such that $g/a = 32$. Since $W = 30{,}000$ and $\gamma = 3$, we have

$$\frac{T_0}{\gamma} = \frac{5{,}000 \sin 2\phi_1''}{(\sec \phi_1'' + \tan \phi_1'')}.$$

By successive trials we find that $\phi_2'' = 63^\circ\ 3'\ 14''$, $\phi_1'' = 34^\circ\ 15'\ 15''$ and $T_0/\gamma = 4{,}560.66$. These values lead to $x_1'' = 2{,}934.3$, $y_1'' = 969.9$, $x_2'' = 6{,}664.3$, $y_2'' = 5{,}675.6$. On

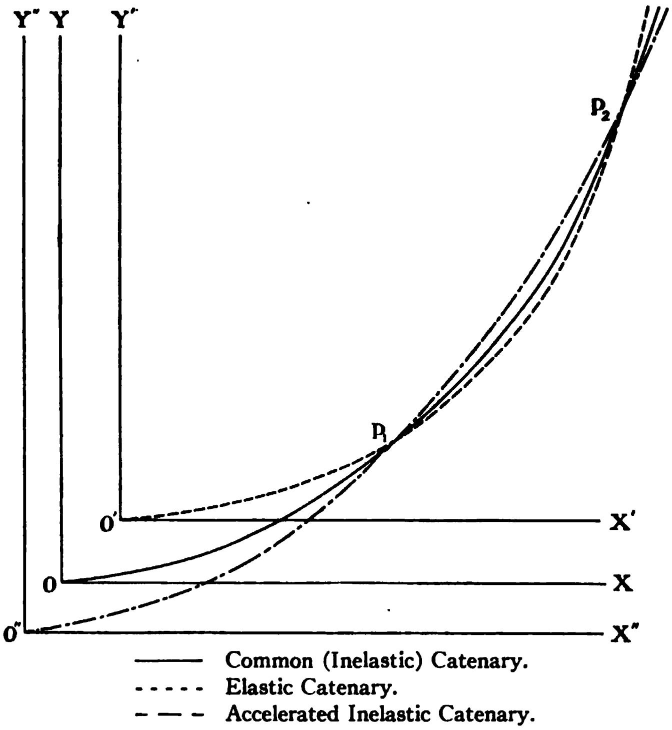


FIG. 4

comparing these coördinates with those of P_1 and P_2 with respect to the axes for the common catenary we find that the origin of O'' has the coördinates $(-\ 5.9,\ -\ 13.9)$ with respect to the x and y axes.

The comparison of points on the common inelastic catenary with those on the accelerated inelastic catenary having the same abscissas is shown in Table II, the final column giving the difference of the ordinates. The relative position of the three curves is shown in Fig. 4.

Note. The accelerated elastic catenary is not included in the graphic comparison as the equation is in series form (51) and for the numerical data this series is not rapidly enough convergent to give the ordinates for the number of terms shown. The elastic curve in Part II, B, is not shown, as the value of λ for steel is so large that the variation from the elastic catenary at rest is not sensible graphically.